Observations and Reports (*continued*).

Lucknow:—North-Western Provinces and Oudh Provincial Museum. Minutes of the Managing Committee. 8vo. *Allahabad* 1889. The Museum.

Madras:—Government Observatory. Results of Observations of the Fixed Stars made with the Meridian Circle. 4to. *Madras* 1890. The Observatory.

Mauritius:—Royal Alfred Observatory. Annual Report. 1888. Folio. [1889]; Mauritius Meteorological Results for 1889. Folio. [1890.] The Observatory.

Melbourne:—Observatory. Report. 1890. Folio. *Melbourne*; Monthly Record. July, 1890. 8vo. *Melbourne*. The Observatory.

New Haven:—Observatory of Yale University. Report. 1889–90. 8vo. 1890. The Observatory.

Paris:—Ministère de l'Instruction Publique et des Beaux-Arts. Rapport sur les Observatoires Astronomiques de Province. 2 Parts. 8vo. *Paris* 1889, 1890. The Department.

February 19, 1891.

Sir WILLIAM THOMSON, D.C.L., LL.D., President, in the Chair.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read:—

I. "On the Sensitiveness of the Bridge Method in its Application to Periodic Electric Currents." By LORD RAYLEIGH, Sec. R.S. Received January 17, 1891.

The most favourable conditions in the ordinary measurement of resistance have been investigated by Schwendler* and by O. Heaviside.† It is here proposed to treat the problem more generally, so as to cover the application to conductors endowed with self-induction, or combined with condensers. The receiving instrument may be supposed to be a telephone, which takes the place of the galvanometer

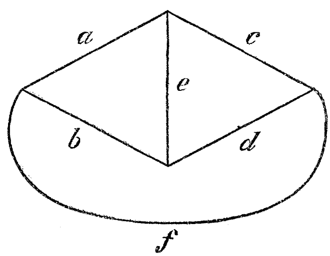
* "On the Galvanometric Resistance to be employed in Testing with Wheatstone's Diagram," 'Phil. Mag.,' vol. 31, p. 364, 1866.

† "On the Best Arrangement of Wheatstone's Bridge for measuring a given Resistance with a given Galvanometer and Battery," 'Phil. Mag.,' vol. 45, p. 114, 1873.

employed in ordinary testing. In the conjugate "battery" branch a periodic electromotive force of given frequency is the origin of the currents.

Special attention will be given to the case where the branches are equal in pairs, *e.g.*, $a = c$, $b = d$ (fig. 1). The advantages of this arrangement are important even in ordinary resistance testing, and in the generalised application are still more to be insisted upon. By mere interchange of a and c and combination of results, the equality of b and d can be verified independently of the exactitude of the ratio $a : c$.

FIG. 1.



If any element in the combination, for example a, be a mere resistance, the difference of potentials at its terminals (V) is connected with the current, x, by the relation

$$V = ax.$$

We have, however, to suppose that a is not merely a resistance or even combination of such. It may include an electromagnet,* and it may be interrupted by a condenser. So long as the current is strictly harmonic, proportional to e^{ipt}, the most general possible relation between V and x is expressed by

$$V = (a_1 + ia_2)\,x,$$

where a_1 and ia_2 are the real and imaginary parts of a complex coefficient a, and are functions of the frequency $p/2\pi$. In the particular case of a simple conductor, endowed with inductance L, a_1 represents the resistance, and a_2 is equal to pL. In general, a_1 is positive; but a_2 may be either positive, as in the above example, or negative. The latter case arises when a resistance, R, is interrupted by a condenser of capacity C. Here $a_1 = R$, $a_2 = -1/pC$. If there be also inductance L,

$$a_1 = R, \qquad a_2 = pL - 1/pC.$$

* An electromagnet here denotes a conductor with sensible inductance. Iron may be present if the range of magnetisation be small.—'Phil. Mag.,' March, 1887.

Since the parts of a_2 may be either positive or negative, there is nothing to hinder its evanescence by compensation. In the above combination of an electromagnet and condenser compensation occurs when $p^2\mathrm{LC} = 1$, that is, when the natural period with terminals connected coincides with the forced period. The combination is then equivalent to a simple resistance;* but a variation of frequency will give rise to a positive or negative a_2.

The case of two electromagnets in parallel is treated in my paper on "Forced Harmonic Oscillations;"† and other combinations have been discussed by Mr. Heaviside and myself. But the above examples will suffice to illustrate the principle that the relation of V to x is one of proportionality, and may be expressed by the single complex symbol a. We fall back at any time upon the case of mere resistance by supposing a to be real. In like manner b, c, d, e, and f are symbols expressing the electrical properties of the remaining branches.

In all electrical problems the generalised quantities a, b, &c., combine, just as they do when they represent simple resistances. Thus, if a, a' be two complex quantities representing two conductors in series, the corresponding quantity for the combination is $a+a'$. Again, if a, a' represent two conductors in parallel, the reciprocal of the resultant is given by addition of the reciprocals of a, a'. For, if the currents be x and x', corresponding to a difference of potentials V at the common terminals,

$$\mathrm{V} = ax = a'x',$$

so that

$$x+x' = \mathrm{V}(1/a+1/a').$$

The investigation of the currents in networks of conductors is usually treated by "Kirchhoff's rules," and this procedure may of course be adopted in the present case to determine the current through the bridge of a Wheatstone combination. But it will be more instructive to put the argument in the form applicable to the forced vibrations of all mechanical systems which oscillate about a configuration of equilibrium.

If $p/2\pi$ represent the frequency of the vibration, the coordinates ψ_1, ψ_2, ψ_3.... determining the condition of the system, and the corresponding forces Ψ_1, Ψ_2, Ψ_3.... are all proportional to e^{ipt}, and the coordinates are linear functions of the forces.‡ For the present purpose we suppose that all the forces vanish, except the first and second. Thus ψ_1, ψ_2 are linear functions of Ψ_1 and Ψ_2, and, conversely, Ψ_1, Ψ_2 may be regarded as linear functions of ψ_1 and ψ_2. We may therefore set

* 'Theory of Sound,' § 46, Macmillan, 1877.

† 'Phil. Mag.,' May, 1886.

‡ 'Theory of Sound,' vol. 1, § 107.

$$\left.\begin{aligned}\Psi_1 &= A\psi_1 + B\psi_2 \\ \Psi_2 &= B\psi_1 + C\psi_2\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

the coefficient of ψ_2 in the first equation being identical with that of ψ_1 in the second by the reciprocal property. The three constants A, B, C are in general complex quantities, functions of p.

In the application that we have to make of these equations, ψ_1, ψ_2, Ψ_1, Ψ_2 will represent respectively currents and electromotive forces in the battery and telephone branches of the combination. The reciprocal property may then be interpreted as follows:—If $\Psi_2 = 0$,

$$B\psi_1 + C\psi_2 = 0,$$

and

$$\psi_2 = \frac{B}{B^2 - AC}\Psi_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

In like manner, if we had supposed $\Psi_1 = 0$, we should have found

$$\psi_1 = \frac{B}{B^2 - AC}\Psi_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

showing that the ratio of the current in one branch to an electromotive force operative in the other is independent of the way in which the parts are assigned to the two branches.

We have now to determine the constants A, B, C in terms of the electrical properties of the system. If ψ_2 be maintained zero by a suitable force Ψ_2, the relation between ψ_1 and Ψ_1 is $\Psi_1 = A\psi_1$. In our application, A therefore denotes the (generalised) resistance to an electromotive force in the battery branch, *when the telephone branch is open*. This resistance is made up of f, the resistance in the battery branch, and of that of the conductors $a+c$, $b+d$ combined in parallel. Thus,

$$A = f + \frac{(a+c)\,(b+d)}{a+b+c+d} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

In like manner,

$$C = e + \frac{(a+b)\,(c+d)}{a+b+c+d} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4').$$

To determine B let us consider the force Ψ_2 which must act in e in order that the current through it (ψ_2) may be zero, in spite of the operation of Ψ_1. We have $\Psi_2 = B\psi_1$. The total current ψ_1 flows partly along the branch $a+c$, and partly along $b+d$. The current through $a+c$ is

$$\frac{\dfrac{1}{a+c}}{\dfrac{1}{a+c} + \dfrac{1}{b+d}}\psi_1 = \frac{(b+d)\,\psi_1}{a+b+c+d} \quad \ldots\ldots\ldots\ldots \quad (5),$$

and that through $b+d$ is

$$\frac{(a+c)\,\psi_1}{a+b+c+d} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6).$$

The difference of potentials at the terminals of e, supposed to be interrupted, is thus

$$\frac{c\,(b+d)\,\psi_1 - d\,(a+c)\,\psi_1}{a+b+c+d};$$

or

$$B = \frac{bc-ad}{a+b+c+d} \quad \ldots\ldots\ldots\ldots\ldots \quad (7).$$

By (4), (4′), (7) the relationship of Ψ_1, Ψ_2 to ψ_1, ψ_2 is completely determined.

The problem of the bridge requires the determination of the current ψ_2, as proportional to Ψ_1, when $\Psi_2 = 0$, that is, when no electromotive force acts in the bridge itself, and the solution is given at once by simple introduction into (2) of the values A, C, B from (4), (4′), (7).

If there be an approximate balance, the expression simplifies. For $bc-ad$ is then small, and B^2 may be neglected relatively to AC in the denominator of (2). Thus, as a sufficient approximation in this case, we have

$$\psi_2/\Psi_1 = \frac{\dfrac{ad-bc}{a+b+c+d}}{\left\{e+\dfrac{(a+b)\,(c+d)}{a+b+c+d}\right\}\left\{f+\dfrac{(a+c)\,(b+d)}{a+b+c+d}\right\}} \quad \ldots \quad (8),$$

in agreement with the equation used by Mr. Heaviside for simple resistances.

The following interpretation of the process leads very simply to the approximate form (8), and may be acceptable to readers less familiar with the general method. Let us first inquire what E.M.F. is necessary in the telephone branch to stop the current through it. If such a force acts, the conditions are, externally, the same as if the branch were open, and the current ψ_1 in the battery branch due to an E.M.F. equal to Ψ_1 in that branch is Ψ_1/A, where A is written for brevity as representing the right-hand member of (4). The difference of potential at the terminals of e, still supposed to be open, is found at once when ψ_1 is known. It is equal to

$$c\times(5) - d\times(6) = B\psi_1,$$

where B is defined by (7). In terms of Ψ_1 the difference of potentials is thus $B\Psi_1/A$. If e be now closed, the same fraction expresses the E.M.F. necessary in e in order to prevent the generation of a current in that branch.

The case that we have to deal with is when Ψ_1 acts in f, and there is no E.M.F. in e. We are at liberty, however, to suppose that two opposite forces, each of magnitude $B\Psi_1/A$, acts in e. One of these, as we have seen, acting in conjunction with Ψ_1 in f, gives no current in e; so that, since electromotive forces act independently of one another, the actual current in e, closed without internal E.M.F., is simply that due to the other component. The question is thus reduced to the determination of the current in e due to a given E.M.F. in that branch.

So far the argument is rigorous; but we will now suppose that we have to deal with an approximate balance. In this case an E.M.F. in e gives rise to very little current in f, and in calculating the current in e we may suppose f to be broken. The total resistance to the force in e is then given simply by C of equation (4′), and the approximate value for ψ_2 is derived by dividing $-B\Psi_1/A$ by C, as we found in (8).

A continued application of the foregoing process gives ψ_2/Ψ_1 in the form of an infinite geometric series:—

$$\psi_2/\Psi_1 = -\frac{B}{AC}\left\{1+\frac{B^2}{AC}+\frac{B^4}{A^2C^2}+\ldots\right\} = \frac{B}{B^2-AC} \quad \ldots \quad (2).$$

This is the rigorous solution already found; but the first term of the series suffices for practical purposes.

The form of (8) enables us at once to compare the effects of increments of resistance and inductance in disturbing a balance. For let $ad = bc$, and then change d to $d+d'$ where $d' = d'_1+id'_2$. The value of ψ_2/Ψ_1 is proportional to d', and the amplitude of the vibratory current in the bridge is proportional to Mod d', that is, to $\sqrt{(d'^2_1+d'^2_2)}$. Thus d'_1, d'_2 are equally efficacious when numerically equal.

The next application that we shall make of (8) is to the generalised form of Schwendler's problem. When all else is given, how should the telephone, or other receiving instrument, be wound in order to get the greatest effect?

If by separation of real and imaginary parts we set

$$e = e_1+ie_2, \qquad \frac{(a+b)(c+d)}{a+b+c+d} = r_1+ir_2 \quad \ldots\ldots \quad (9),$$

the factor in the denominator of (6) with which we are concerned becomes

$$e_1+r_1+i(e_2+r_2);$$

and the square of the modulus is given by

$$\text{Mod}^2 = (e_1+r_1)^2+(e_2+r_2)^2 \quad \ldots\ldots\ldots\ldots \quad (10).$$

In this equation e_1, r_1 are essentially positive, while e_2, r_2 may be either positive or negative. If e_1 and e_2 are both at disposal, the minimum of (10), corresponding to the maximum current, is found by making

$$e_1 = 0, \qquad e_2 = -r_2 \quad \text{............} \quad (11).$$

But this is not the practical question. As in the case of simple resistances, what we have to aim at is not to render the current in the bridge a maximum, but rather the *effect* of the current. Whether the receiving instrument be a galvanometer or a telephone, we cannot in practice reduce its resistance to zero without at the same time nullifying the effect desired. We must rather regard the space available for the windings as given, and merely inquire how it may best be utilised. Now the effect required to be exalted is, *cœteris paribus*, proportional to the number of windings (m); and, if the space occupied by insulation be proportional to that occupied by copper, the resistance varies as m^2. So also does the inductance; and accordingly, if the instrument be connected to the bridge by leads sensibly devoid of resistance and inductance,

$$e_1 + ie_2 = m^2(\epsilon_1 + i\epsilon_2) \quad \text{................} \quad (12),$$

where ϵ_1, ϵ_2 are independent of m. The quantity whose modulus is to be made a minimum by variation of m is thus

$$\frac{e_1 + ie_2 + r_1 + ir_2}{m} = \frac{r_1 + m^2\epsilon_1 + i(r_2 + m^2\epsilon_2)}{m};$$

and we have

$$\text{Mod}^2 = \frac{(r_1 + m^2\epsilon_1)^2 + (r_2 + m^2\epsilon_2)^2}{m^2}$$

$$= (r_1^2 + r_2^2)\, m^{-2} + 2\,(r_1\epsilon_1 + r_2\epsilon_2) + (\epsilon_1^2 + \epsilon_2^2)m^2.$$

This is a minimum by variation of m when

$$m^4 = \frac{r_1^2 + r_2^2}{\epsilon_1^2 + \epsilon_2^2},$$

or

$$\text{Mod}\,(r_1 + ir_2) = \text{Mod}\,(e_1 + ie_2) \quad \text{..........} \quad (13).$$

We may express this result by saying that to get the best effect the instrument must be so wound that its *impedance* is equal to that of the compound conductor $r_1 + ir_2$. If for any reason the inductances can be omitted from consideration, then the resistance of the instrument is to be made equal to r_1, in accordance with Schwendler's rule.

The case of the "battery" branch may often be treated in like manner. As Mr. Heaviside has shown, if a number of cells are

available for ordinary resistance testing, they should be combined, so that their resistance is equal to that (s_1) of the corresponding combination of wires in parallel. Periodic currents may be conceived to arise from the rotation of a coil in a magnetic field of given strength. If the space occupied by the windings of the coil be supposed to be given, their number m will be determined by the condition of equal impedances. Thus, if

$$\frac{(a+c)(b+d)}{a+b+c+d} = s_1 + is_2 \quad \ldots\ldots\ldots\ldots\ldots \quad (14),$$

$$\text{Mod}\,(f_1+if_2) = \text{Mod}\,(s_1+is_2) \quad \ldots\ldots\ldots\ldots \quad (15),$$

in analogy with (13).

The above is the solution of the problem, if the coils of the sending and receiving instruments represent the whole of their respective branches, and are limited to occupy given spaces. The inductances and resistances cannot then be varied independently. But there would often be no difficulty in escaping from this limitation. The inclusion of additional resistance, external to the instrument, can only do harm; but the case is otherwise with inductance, positive or negative. If the inductance of the instrument added to r_2, or to s_2, be positive, the total inductance may be reduced to zero by the insertion of a suitable condenser, and this without material increase of resistance. If the inductance be already negative, the remedy is not so easily carried out; but, theoretically, it is possible to add the necessary inductance without sensible increase of resistance. The greater the frequency of vibration, the more feasible does this course become. We may, therefore, without much violence, suppose that the inductances of two branches can be reduced to zero without additional resistance. Thus,

$$e_2+r_2 = 0, \qquad f_2+s_2 = 0 \quad \ldots\ldots\ldots\ldots \quad (16);$$

and the condition of maximum efficiency of the transmitting and receiving coils is then given by Schwendler's rule,

$$e_1 = r_1, \qquad f_1 = s_1 \quad \ldots\ldots\ldots\ldots\ldots \quad (17).$$

These suppositions form a reasonable basis for further investigation; but conclusions founded upon them will be subject to re-examination, especially in extreme cases. We may also now introduce the promised simplification,

$$a = c, \qquad b = d \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (18),$$

in accordance with which (8) becomes

$$\psi_2/\Psi_1 = \frac{d-b}{4b}\,\frac{2ab/(a+b)}{\{e+\frac{1}{2}(a+b)\}\,\{f+2ab/(a+b)\}} \quad \ldots\ldots \quad (19).$$

Also $$r_1+ir_2 = \tfrac{1}{2}(a+b) = \tfrac{1}{2}(a_1+b_1)+\tfrac{1}{2}i(a_2+b_2) \quad \ldots\ldots \quad (20).$$

$$s_1+is_2 = \frac{2(a_1+ia_2)(b_1+ib_2)}{a_1+ia_2+b_1+ib_2}$$

$$= 2\,\frac{(a_1+b_1)(a_1b_1-a_2b_2)+(a_2+b_2)(a_2b_1+a_1b_2)}{(a_1+b_1)^2+(a_2+b_2)^2}$$

$$+2i\,\frac{(a_1+b_1)(a_2b_1+a_1b_2)-(a_2+b_2)(a_1b_1-a_2b_2)}{(a_1+b_1)+(a_2+b_2)^2} \quad \ldots \quad (21).$$

It may be well to examine, first, the consequences of (19), in the case of simple resistances. Here

$$r_1 = \tfrac{1}{2}(a_1+b_1), \qquad r_2 = 0 \quad \ldots\ldots\ldots \quad (22);$$

$$s_1 = 2a_1b_1/(a_1+b_1), \qquad s_2 = 0 \quad \ldots\ldots\ldots \quad (23).$$

In accordance with the plan proposed, we are to make $e_2 = 0$, $f_2 = 0$;* $e_1 = r$, $f_1 = s_1$. Our equation then becomes

$$\psi_2/\Psi_1 = \frac{d_1-b_1}{8b_1(a_1+b_1)} \quad \ldots\ldots\ldots\ldots\ldots \quad (24).$$

Here a_1 is still at disposal, and we see that according to (24) it ought to be diminished without limit. This conclusion does not harmonize with one obtained by Mr. Heaviside.† It must be observed, however, that $a_1 = 0$ is unpractical, involving, as it does, $s_1 = 0$, $f_1 = 0$. Even according to (24) there is little to be gained by diminishing a_1 below, say, $\tfrac{1}{2}b_1$. In this case

$$a_1 = \tfrac{1}{2}b_1, \qquad e_1 = r_1 = \tfrac{3}{4}b_1, \qquad f_1 = s_1 = \tfrac{2}{3}b_1 \quad \ldots\ldots \quad (25).$$

Such an arrangement as (25) may be recommended for practical use.

When b_1 is large, there may be advantage in taking a_1 relatively smaller than in the above example. In such cases we approach the limiting condition of things, and have approximately

$$e_1 = r_1 = \tfrac{1}{2}b_1, \qquad f_1 = s_1 = 2a_1 \quad \ldots\ldots\ldots\ldots \quad (26),$$

$$\psi_2/\Psi_1 = \frac{d_1-b_1}{8b_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27).$$

And the smallness of f_1 in comparison with b_1 may sometimes be a convenience.

* These conditions require no attention in galvanometric testing with steady currents, being satisfied by $p = 0$, independently of the nature of the instrument.

† *Loc. cit.*, p. 120, "In conclusion, if, to measure a certain resistance, the best resistances for the galvanometer, battery, and the three sides, a, b, c, were required, then we should have to make $a = b = c = d = e = f$."

The next remark that has to be made is that, even when the conductors, b and d, to be compared are endowed with sensible inductances (positive or negative), the problem may still, theoretically, be brought under the above head. Suppose, for example, that b, d represent nearly equal electromagnets. Their inductances may be compensated by the introduction (in series) of suitable equal condensers into these branches, so that b and d are reduced to b_1 and d_1. If then we assume a to be a simple resistance ($a_2 = 0$), the solution is as before. Two objections may here be raised. First, on the theoretical side it has not been proved to be advantageous to assume $a_2 = 0$; and, secondly, the introduction of extraneous condensers,* even with interchange, into the branches to be accurately compared may be a complication unfavourable to success.

We will now resume the consideration of (19), supposing that

$$e = e_1 + ie_2 = r_1 - ir_2, \qquad f = f_1 + if_2 = s_1 - is_2 \quad \ldots\ldots \quad (28),$$

r_1, r_2, s_1, s_2 being given by (20), (21). Thus,

$$\psi_2/\Psi_1 = \frac{d-b}{16}\,\frac{s_1+is_2}{br_1s_1} \quad \ldots\ldots\ldots\ldots\ldots \quad (29),$$

and the question before us is how to make the modulus of the second fraction on the right a maximum by variation of a. In the denominator of this fraction r_1 and s_1 are real, and the modulus of b is $\sqrt{(b_1^2+b_2^2)}$. For the numerator we have

$$\frac{1}{a}+\frac{1}{b} = \frac{1}{a_1+ia_2}+\frac{1}{b_1+ib_2} = \frac{2}{s_1+is_2} = \frac{2(s_1-is_2)}{s_1^2+s_2^2},$$

so that

$$\frac{2s_1}{s_1^2+s_2^2} = \frac{a_1}{a_1^2+a_2^2}+\frac{b_1}{b_1^2+b_2^2}.$$

Also from the definition of s

$$s_1^2+s_2^2 = \frac{4(a_1^2+a_2^2)(b_1^2+b_2^2)}{(a_1+b_1)^2+(a_2+b_2)^2};$$

so that

$$\frac{s_1^2}{s_1^2+s_2^2} = \frac{(a_1^2+a_2^2)(b_1^2+b_2^2)}{(a_1+b_1)^2+(a_2+b_2)^2}\left\{\frac{a_1}{a_1^2+a_2^2}+\frac{b_1}{b_1^2+b_2^2}\right\}^2.$$

Thus

$$\text{Mod}\,\frac{br_1s_1}{s_1+is_2} = \frac{(a_1+b_1)\{a_1(b_1^2+b_2^2)+b_1(a_1^2+a_2^2)\}}{2\sqrt{(a_2^2+a_2^2)}\,.\,\sqrt{\{(a_1+b_1)^2+(a_2+b_2)^2\}}} \quad \ldots \quad (30),$$

and this is to be made a minimum by variation of a_1, a_2.

* The use of condensers or electromagnets in the branches e and f stands, of course, upon a different footing.

We shall show presently that (30) can be reduced to zero; but for the moment we will so far limit the generality of a_1, a_2 as to suppose that $a_1 = xb_1$, $a_2 = xb_2$, x being real and positive.

(30) then reduces to $\frac{1}{2}b_1^2(1+x)$; and by (29)

$$\text{Mod}\ \psi_2/\Psi_1 = \frac{\text{Mod}\ (d-b)}{8b_1^2(1+x)} \ldots\ldots\ldots\ldots \quad (31).$$

Accordingly, the maximum sensitiveness cannot be attained until x is reduced to zero, so that a_1, a_2 vanish. (31) may be regarded as a generalised form of (24), free from the limitation that $b_2 = 0$, provided a_2 be so taken that $a_2/b_2 = a_1/b_1$.

We will now suppose in (30) that a_1 and a_2 are both small, and in the first instance that b_1 is finite. We have

$$\tfrac{1}{2}b_1\sqrt{(b_1^2+b_2^2)}\ \frac{a_1}{\sqrt{(a_1^2+a_2^2)}}+\frac{\frac{1}{2}b_1^2}{\sqrt{(b_1^2+b_2^2)}}\ \sqrt{(a_1^2+a_2^2)} \ldots (32);$$

and this reduces ultimately to its first term, depending upon the *ratio* only of a_1 and a_2. The expression vanishes if $a_1 : a_2$ be small enough, so that (30) can certainly be thus reduced to zero. It is remarkable that the expression for the sensitiveness should be capable of becoming infinite by suitable choice of a_2. If we first suppose that a_2 is absolutely zero, and afterwards that a_1 diminishes without limit, the ultimate value of (32) is $\frac{1}{2}b_1\sqrt{(b_1^2+b_2^2)}$, in place of zero.

From the practical point of view, these conclusions from our equations are not particularly satisfactory. We began with certain proposals which, in ordinary cases, could be carried out; but in the end we are directed to apply them to an extreme and impossible state of things. We have found, however, in what direction we must tend in the search for sensitiveness; and useful information may be gathered from (32). In practice a_1 could not be reduced below a certain point. The question may then be asked, what is the best value of a_2, when a_1 is given? From (32) we find at once that

$$a_1^2+a_2^2 = \frac{a_1(b_1^2+b_2^2)}{b_1} \ldots\ldots\ldots\ldots\ldots \quad (33),$$

(32) then becoming

$$b_1\sqrt{(a_1b_1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (34).$$

In this case from (29)

$$\text{Mod}\ \psi_2/\Psi_1 = \frac{\text{Mod}\ (d-b)}{16b_1\sqrt{(a_1b_1)}} \ldots\ldots\ldots\ldots \quad (35),$$

independent of b_2.

If we suppose in (32) that $a_2 = 0$, we have

$$\tfrac{1}{2}b_1\sqrt{(b_1^2+b_2^2)}+\frac{\frac{1}{2}b_1^2a_1}{\sqrt{(b_2^2+b_2^2)}} \ldots\ldots\ldots\ldots \quad (36).$$

To take a numerical example, let $b_2 = 0$; and suppose $a_1 = \frac{1}{10} b_1$. Then, according to (33), $a_2 = \pm \frac{3}{10} b_1$. Also by (20), (21),

$$e_1 = \tfrac{11}{20} b_1, \qquad e_2 = \mp \tfrac{3}{20} b_1;$$
$$f_1 = \tfrac{4}{13} b_1, \qquad f_2 = \mp \tfrac{6}{13} b_1.$$

The corresponding minimum value of (32), equal to (34), is $b_1^2/\sqrt{(10)}$.

But with this value of a_1 the gain by allowing a_2 to be finite is not great. If $a_2 = 0$,

$$e_1 = \tfrac{11}{20} b_1, \qquad e_2 = 0;$$
$$f_2 = \tfrac{11}{65} b_1, \qquad f_2 = 0;$$

and the value of (32), equal to (36), is $\frac{11}{20} b_1^2$.

We see from (36) that when $a_2 = 0$ there is little to be gained by further reduction of a_1. But when a_2 is suitably chosen the gain may be worth having. Thus, in (34), if $a_1 = \frac{1}{100} b_1$, we have $\frac{1}{10} b_1^2$. Corresponding to this $a_2 = \pm \frac{1}{10} b_1$ nearly, and

$$e_1 = \tfrac{1}{2} b_1, \qquad e_2 = \mp \tfrac{1}{20} b_1;$$
$$f_1 = \tfrac{1}{25} b_1, \qquad f_2 = \mp \tfrac{1}{5} b_1.$$

These are not unreasonable proportions, and we see that the use of a_2 may be advantageous, even when the subject of measurement is a mere resistance. It will be remarked too that, except as regards e_2, f_2, the sign of a_2 is immaterial.

When the branches b, d consist of electromagnets, and still more when they consist of condensers, b_1 may be very small. If we suppose it to be zero, (30) becomes

$$\frac{a_1^2 b_2^2}{2\sqrt{(a_1^2+a_2^2)} \cdot \sqrt{\{a_1^2+(a_2+b_2)^2\}}} \quad \ldots\ldots\ldots\ldots \quad (37).$$

Corresponding to this from (20), (21),

$$e_1 = \tfrac{1}{2} a_1, \qquad e_2 = -\tfrac{1}{2}(a_2+b_2) \quad \ldots\ldots \quad (38),$$

$$f_1 = \frac{2a_1 b_2^2}{a_1^2+(a_2+b_2)^2}, \qquad f_2 = -\frac{2a_1^2 h_2 + 2a_2 b_2(a_2+b_2)}{a_1^2+(a_2+b_2)^2} \quad .. \quad (39).$$

From (37) we see that the increase of a_2 is favourable, especially if the sign be the same as of b_2. Even if $a_2 = 0$, (37) now assuming the form

$$\frac{a_1 b_2^2}{2\sqrt{(a_1^2+b_2^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

can be reduced to zero by taking a_1 small enough. But of course (37) ceases to be applicable unless b_1 be small relatively to a_1. In correspondence with (40),

$$e_1 = \tfrac{1}{2}a_1, \qquad e_2 = -\tfrac{1}{2}b_2 \quad \ldots\ldots\ldots \quad (41);$$

$$f_1 = \frac{2a_1b_2^2}{a_1^2+b_2^2}, \qquad f_2 = -\frac{2a_1^2b_2}{a_1^2+b_2^2} \quad \ldots\ldots \quad (42).$$

As an example of (37), suppose

$$a_1 = \tfrac{1}{4}b_2, \qquad a_2 = 4b_2.$$

Then
$$(37) = \frac{b_2^2}{640} \text{ nearly.}$$

Also approximately

$$e_1 = \tfrac{1}{8}b_2, \qquad e_2 = -\tfrac{5}{2}b_2, \qquad f_1 = \tfrac{1}{50}b_2, \qquad f_2 = -\tfrac{8}{5}b_2.$$

If b_2 represent the stiffness of a condenser, f_2 must be a positive inductance, and its magnitude, relatively to f_1, would probably constitute a difficulty.

As an example, with a_2 equal to zero, take

$$a_1 = \tfrac{1}{10}b_2, \qquad a_2 = 0.$$

Then
$$(37) = (40) = \tfrac{1}{20}b_2^2 \text{ nearly,}$$

and

$$e_1 = \tfrac{1}{20}b_2, \qquad e_2 = -\tfrac{1}{2}b_2, \qquad f_1 = \tfrac{1}{5}b_2, \qquad f_2 = -\tfrac{1}{50}b_2.$$

So far as the general theory is concerned, it is a matter of indifference whether the indicating instrument be in the branch e, or in f. The latter corresponds to the connections in De Sauty's method of testing condensers by means of the galvanometer. In practice, more space would probably be available for the coils of a transmitting instrument than of the receiving instrument, at least, if the latter be a telephone; and this would tell in favour of choosing that branch for the transmitter which should have the larger time constant (L/R).

To get an idea of the relative capacities, resistances, and inductances involved, we must assume a particular pitch. A frequency suitable for telephonic experiments is 1000 per second, for which $p = 2000\pi$. Thus, if the value of a_2 for a condenser of capacity C, and for an inductance L, and that of a_1 for a resistance R, are all numerically equal,

$$R = 2000\pi L = \frac{1}{2000\pi C}.$$

If R be 1 ohm, equal to 10^9 C.G.S., the corresponding capacity is $1{\cdot}6 \times 10^{-13}$ C.G.S., equal to 160 microfarads, and the corresponding

inductance is $1{\cdot}6\times10^5$ C.G.S. Again, if C be one microfarad, equal to 10^{-15} C.G.S., R is 160 ohms, and L is $2{\cdot}5\times10^7$ cm.

In the preceding calculations e and f are supposed to be adjusted to the values most favourable to the effect in the receiving instrument. A question, which arises quite as often in practice, is how to make the best of given instruments. The full answer is necessarily somewhat complicated; for there could be no objection to the insertion of a condenser for example, if the sensitiveness could be improved thereby. In what follows, however, the transmitting and receiving branches will be supposed to be fully given, so that e and f are known complex quantities; and the only question to be considered is as to the most suitable value of a, assumed to be equal to c.

For this purpose the modulus of the second fraction on the right in (19) is to be a maximum, or that of

$$(a+b+2e)\left(\frac{1}{a}+\frac{1}{b}+\frac{2}{f}\right) \quad\ldots\ldots\ldots\ldots\ldots \quad (43)$$

is to be a minimum, by variation of a. The problem thus arising of determining the minimum modulus of a function of a complex quantity may be treated generally.

Let

$$\mathrm{F}(z) = \mathrm{F}(x+iy) = \phi(x,y)+i\psi(x,y),$$

and let it be required to find when the modulus² of F (z), viz., $\phi^2+\psi^2$, is a minimum by variation of x,y. We have

$$\phi\frac{d\phi}{dx}+\psi\frac{d\psi}{dx}=0, \qquad \phi\frac{d\phi}{dy}+\psi\frac{d\psi}{dy}=0 \quad\ldots\ldots \quad (44).$$

And in general

$$\frac{d\phi}{dx}=\frac{d\psi}{dy}, \qquad \frac{d\phi}{dy}=-\frac{d\psi}{dx} \quad\ldots\ldots \quad (45).$$

In order that (44), (45) may both obtain, we must have either $\phi^2+\psi^2=0$, or else

$$\frac{d\phi}{dx}=0, \qquad \frac{d\phi}{dy}=0, \qquad \frac{d\psi}{dx}=0, \qquad \frac{d\psi}{dy}=0.$$

The latter conditions are equivalent to

$$\mathrm{F}'(z)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (46).$$

For example, let

$$\mathrm{F}(z) = (z+\alpha)\left(\frac{1}{z}+\beta\right) \quad\ldots\ldots\ldots\ldots\ldots \quad (47),$$

where α, β are complex constants.

The application of (46) gives

$$z^2 = \alpha/\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (48),$$

and

$$F(z) = \{1+\sqrt{(\alpha\beta)}\}^2 \quad \ldots\ldots\ldots\ldots\ldots \quad (49).$$

We see then that the modulus of (43) will be a minimum, when

$$a^2 = \frac{b+2e}{2/f+1/b} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (50),$$

and in taking the square root the ambiguity must be so determined as to make the real part of a positive.

Equation (50) coincides with that obtained by Mr. Heaviside for the case where all the quantities are real.

II. "On the Influence of Pressure on the Spectra of Flames." By G. D. Liveing, M.A., F.R.S., Professor of Chemistry, and J. Dewar, M.A., F.R.S., Jacksonian Professor, University of Cambridge. Received January 22, 1891.

We have already described ('Phil. Trans.,' A, 1888) the remarkable spectrum of the oxy-hydrogen flame burning at the ordinary atmospheric pressure. Recently we have examined the spectrum of the same flame at various pressures: hydrogen burning in excess of oxygen up to a pressure of 40 atmospheres, and oxygen in excess of hydrogen up to a pressure of 25 atmospheres, also that of the mixed gases burning in carbonic acid gas.

The apparatus employed was an adaptation of one of the tubes used in our experiments on the absorption spectra of compressed gases ('Phil. Mag.,' September, 1888, and 'Roy. Soc. Proc.,' vol. 46, p. 222). It consisted of a steel cylinder, about 50 mm. in internal diameter and 225 mm. long, fitted at one end with a quartz stopper, a, in the annexed figure, and with a jet, b, for burning the gas, adapted by a properly fitting union joint to the opposite end. There were two tubes, c and d, connected to the cylinder at the sides, of which one, c, served for the introduction of gas, while the other, d, was fitted with a stopcock and was used to draw off the water formed, or to reduce the pressure of the gas in the cylinder if that was desired. The flame was observed, nearly end on, through the quartz stopper. The whole apparatus was kept cool by a stream of cold water running on to a sponge cloth wrapped round the cylinder. In the course of the tube conveying gas to the jet b was interposed a small cylinder, e, in which sodium was placed, and by heating this, the gas entering could be charged with sodium vapour.